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Skin Friction and Velocity Profile Family for Compressible Turbulent Boundary Layers

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The paper presents a general approach to constructing mean velocity profiles for compressible turbulent boundary layers with isothermal or adiabatic walls. The theory is based on a density-weighted transformation that allows the extension of the incompressible similarity laws of the wall to the compressible regions. The velocity profile family is compared to a range of experimental data, and excellent agreement is obtained. A self-consistent skin friction law, which satisfies the proposed velocity profile family, is derived and compared with the well-known Van Driest II theory for boundary layers in zero pressure gradient. The results are found to be at least as good as those obtained by using the Van Driest II transformation.

I. Introduction

IN a careful evaluation of compressible turbulent boundary layer data, Fernholz and Finley[1] have concluded that the incompressible law of the wall is preserved when the velocity profile is transformed using Van Driest's extension of the mixing length formula. In compressible flow the usual logarithmic part of law of the wall becomes

$$\frac{U_c}{u_\tau} = \frac{1}{\kappa} \ln y^+ + C \tag{1}$$

where $u_\tau = \sqrt{\tau_w/\rho_w}$; $y^+ = u_\tau y/\nu_w$; $\kappa \approx 0.41$ is the von Kármán constant; C is chosen to be the same as its incompressible flow counterpart, 5.2, and this choice is supported by the experimental data;[1] and finally U_c is the transformed velocity[2] defined by

$$U_c = \int \left(\frac{\rho}{\rho_w}\right)^{1/2} \mathrm{d}U \tag{2}$$

Near a solid surface, convection can be neglected, and if we also assume $\tau = \tau_w$, the energy equation can be integrated with respect to y to give

$$q = q_w + U\tau_w \tag{3}$$

By substituting $q = -(\mu_t c_p/Pr_t)(\partial T/\partial y)$ and $\tau_w = \mu_t(\partial U/\partial y)$ into Eq. (3) and assuming that the effective Prandtl number in the viscous sublayer is equal to Pr_t, one may integrate Eq. (3) with respect to U to obtain

$$T = T_w - \frac{Pr_t q_w U}{c_p \tau_w} - \frac{Pr_t U^2}{2c_p} \tag{4}$$

A change in the effective Prandtl number in the sublayer simply changes the constant of integration, here equal to T_w: Eq. (4) seems to be an adequate fit to data. Equation (4) establishes the relationship between T_w and q_w/τ_w.

Since the pressure in the boundary layer is independent of y, the density ratio appearing in Eq. (2) can be replaced by the temperature ratio, which can be obtained from Eq. (4). The Van Driest transformation yields

$$U_c = \sqrt{B}\left[\sin^{-1}\left(\frac{A+U}{D}\right) - \sin^{-1}\left(\frac{A}{D}\right)\right] \tag{5}$$

where

$$A = q_w/\tau_w$$

$$B = 2c_p T_w/Pr_t$$

$$D = \sqrt{A^2 + B}$$

Or, one may also express the transformation in terms of the inverse of Eq. (5)

$$\frac{U}{u_\tau} = \frac{1}{R}\sin\left(\frac{RU_c}{u_\tau}\right) - H\left[1 - \cos\left(\frac{RU_c}{u_\tau}\right)\right] \tag{6}$$

where

$$R = u_\tau/\sqrt{B}$$

$$H = A/u_\tau$$

In the present work, the turbulent Prandtl number Pr_t is assumed to be 0.9, and Eq. (4) implies that the recovery factor r is also 0.9 because the adiabatic wall temperature is defined by

$$T_{aw} = T_e\,[1 + r\frac{(\gamma - 1)}{2} M_e^2]$$

Equations (5) or (6), with U_c given by Eq. (1), apply only to the log-law region. However, Eqs. (5) or (6) can be used formally throughout the layer, in general giving a U_c profile

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that deviates from Eq. (1) in the outer layer and in the viscous sublayer. To do this, we have applied Coles' law of the wake[3] for the outer regions and included a Van Driest type of mixing-length damping[4] for the viscous sublayer. Equation (1) can therefore be replaced by

$$U_c^+ = \frac{U_c}{u_\tau} = U_{c,b}^+ + \frac{\Pi}{\kappa} w\left(\frac{y}{\delta}\right) \tag{8}$$

Here, the w-function is an assumed wake profile for which we adopted the suggested formula by Coles,[3] $w(\eta) = 2 \sin^2(\eta\Pi/2)$; Π is a wake parameter, which Fernholz and Finley[1] suggested is nearly independent of Mach number if expressed as a function of the empirically-chosen Reynolds number $Re_{\delta 2} = Re_\theta(\mu_e/\mu_w) = \rho_e U_e \theta/\mu_w$ based on the viscosity at the wall; and $U_{c,b}^+$ is a pure law-of-the-wall profile defined by

$$\frac{dU_{c,b}^+}{dy^+} = \frac{2}{[(1 + 4l^{+2})^{1/2} + 1]} \tag{9}$$

where

$$l^+ = \kappa y^+(1 - e^{-y^+/A^+})$$

For $A^+ = 25.53$, it asymptotes to Eq. (1).

Although the viscous sublayer is neglected in Coles' original proposal for Eq. (8), we found that it is important to include the sublayer contribution for high-speed flows because it becomes thicker and may occupy a substantial portion of the whole boundary layer at hypersonic Mach number. However, sublayer data for compressible flows are scarce, especially for large heat transfer rates (large density gradients) where the empirical Van Driest exponential damping function may be inadequate. Therefore, we cannot claim detailed reliability of our profile family in the sublayer, but it is probably adequate for calculations of integral thickness, certainly for evaluating Re_θ. Note also that, because the temperature near the wall is very high for high Mach number flow, $Re_{\delta 2}$ is usually low enough to be in the range where Π depends on Reynolds number in low-speed flow.

With Π as a free parameter, the profile can be used to fit experimental data or as the basis of an "integral" calculation method, both for arbitrary pressure gradients. Once Π is specified in zero pressure gradient, Eq. (8), evaluated at $y = \delta$, gives a skin-friction law. In this work, we have used two ways to define the Π function. The first way is to choose the Π function directly from a curve fit to the data assembled by Coles.[5] Cebeci and Smith[6] recommended the following curve fitting formula:

$$\Pi = 0.55[1 - \exp(-0.24\sqrt{Re_\theta} - 0.298\, Re_\theta)] \tag{10}$$

The second way to derive the Π function is to require Eq. (8), evaluated at $y = \delta$, to agree with a standard skin-friction correlation (in incompressible flow). This of course means that inaccuracies in velocity profile measurement are unimportant, and may be the better approach, particularly if one's main interest is in predicting skin friction rather than velocity profiles. Note also that the accuracy of the measured Π function depends on accurate measurement of the skin friction. Here, we have chosen the von Kármán-Schoenherr skin-friction formula,

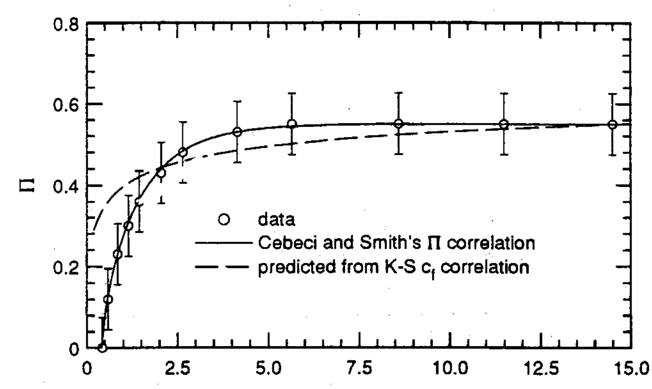
$$c_f = 1/[17.08\,(\log_{10} Re_\theta)^2 + 25.11 \log_{10} Re_\theta + 6.012] \tag{11}$$

The comparison of the Π functions with experimental data is shown in Fig. 1a. The Π function predicted using the von Kármán-Schoenherr skin friction correlation seems to fall within the experimental data bounds for Reynolds numbers larger than 1000. Figure 1b shows the variations of skin friction c_f with momentum thickness Reynolds number Re_θ in an incompressible zero-pressure-gradient boundary-layer flow. It can be seen that the c_f prediction using Π defined by Eq. (10) agrees very closely with the von Kármán-Schoenherr correlation for $Re_\theta > 1000$. For $Re_\theta < 1000$, the prediction gives higher c_f than the correlation, corresponding to the lower values of Π in the experimental data. But drawing firm conclusions is difficult because the difference between the present prediction and the von Kármán-Schoenherr correlation is within the likely error of the data. It is interesting to note that at $Re_\theta = 300$, the von Kármán-Schoenherr correlation gives almost the same skin-friction value as the direct numerical simulation of Spalart.[7] However, Spalart (private communication) points out that at this low Reynolds number the simulation, just like an experiment, may still be affected by the upstream conditions. The predicted velocity profile at $Re_\theta = 300$ also agrees very well with the simulation, as shown in Fig. 2. It should be noted that, at this Re_θ, the wake component in the buffer layer is significant, giving rise to a profile slope in the log-law region larger than $1/\kappa$. It is of course fortuitous that Coles' "$\sin^2$" wake profile gives good agreement with the simulation in this region: the "law of the wake" is a curve fit and does not really contain any physics.

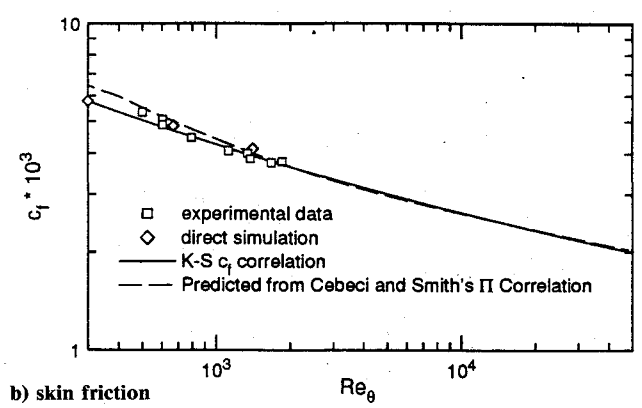
When extending Eq. (8) to compressible flow, we have found that except for Watson's[8] Mach 11 helium flow case, in which $Re_{\delta 2}$ is less than 1000, the two Π functions produce almost identical results. Thus, unless otherwise stated, only results obtained using the Π function satisfying the von Kármán-Schoenherr correlation will be reported in the following.

Hopkins and Inouye[9] concluded that the Van Driest II skin-friction formula is the best of those they compared. Since then, the formula has been used as a benchmark to compare compressible turbulence models. Van Driest II is based on the assumption that the mixing length follows the Kármán hypothesis

$$l = \kappa \left| \frac{\partial U/\partial y}{\partial^2 U/\partial y^2} \right| \tag{12}$$



a) Π function



b) skin friction

Fig. 1 Comparison of the two Π functions. Error bars derived from data plotted by Coles.[5]

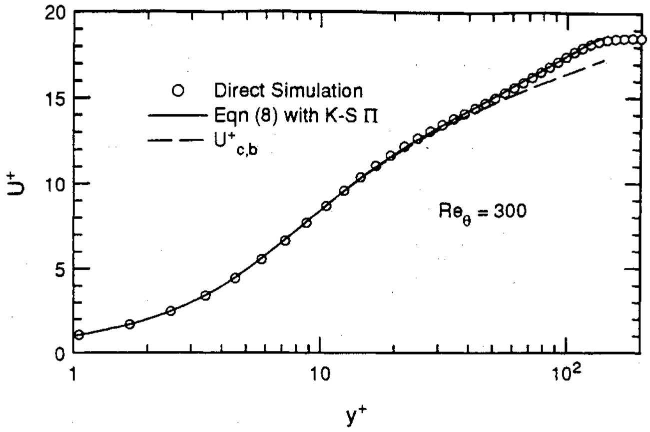


Fig. 2 Mean velocity profiles at $Re_\theta = 300$, compared with direct numerical simulation.

In the log-law region of an incompressible boundary layer, Eq. (12) reduces to Prandtl mixing length, $l = \kappa y$. But, for a compressible boundary layer, it can be shown that the Kármán length-scale assumption leads to a log region for variables U_c^+ and y_c^+, where U_c^+ is defined by Eq. (2) and y_c^+ is a transformed coordinate defined as

$$y_c^+ = \int \left(\frac{\rho}{\rho w}\right)^{1/2} dy^+ \tag{13}$$

In contrast, the experimental evidence suggests a log-law relation between U_c^+ and y^+, not y_c^+, indicating that the Prandtl mixing length formula still holds even for high Mach number flows. This observation is also reflected in the work of Coakley and Huang[10] in evaluating several turbulence models for hypersonic flows. They found that under the strongly cooled wall conditions, models that predict skin friction in agreement with Van Driest II tend not to follow the compressible law of the wall, and the models that predict lower skin friction than Van Driest II do better on velocity profiles. There is clearly an inconsistency between the Van Driest II theory and the compressible law of the wall outlined above. For a more extensive theoretical discussion of law-of-the-wall/law-of-the-wake matching in compressible flow see Barnwell,[11] who uses the Prandtl formulation.

Bradshaw[12] secured good agreement with Van Driest II skin friction by allowing the constant C in Eq. (1) to vary according to the frictional Mach number and the wall heat transfer parameter, u_τ/c_w and $q_w/(\rho_w c_p u_\tau T_w)$, respectively. This approach, termed "Van Driest III" by Bradshaw, can be considered as a reconciliation of the law of the wall and the Van Driest II theory. Unfortunately, the value of C would have to increase from 5.2 to about 6.4 for a Mach 5 flow over an insulated plate at $Re_\theta = 10{,}000$. This drastic change of C is not supported by the experimental evidence in Fernholz and Finley's review. Another way to force the compressible law of the wall to predict Van Driest II skin friction is to alter the strength of the wake component. For a Mach 10 flow over an insulated plate and a moderate small Reynolds number, $Re_{\delta_2} = 1700$ (corresponding to $Re_\theta = 10{,}000$ at room temperature), it can be shown that to satisfy the law of the wall and at the same time be able to match the Van Driest II prediction, Π in Eq. (8) has to be 0.73—a value much larger than the incompressible asymptotic value. This again is not supported by the experimental evidence shown in Fernholz and Finley.[1]

The Van Driest II theory was based on very weak physical arguments. Its popularity is entirely rooted in its success in correlating experimental data, traceable at least as far back as 1971. Since there is an inconsistency between the theory and the compressible law of the wall, a new approach, deviating from Van Driest II theory but still securing a good skin-friction prediction, should perhaps be considered. Such an approach is outlined in the following.

II. Skin Friction Algorithm

The foundation of the present approach is derived from the experimental evidence that the law of the wall and the law of the wake are transferable from incompressible flow to compressible flow, provided that the velocity is defined by the density-weighted transformation, Eq. (5), and the wake parameter Π is correlated with Re_{δ_2}. In other words, Eq. (8) is considered as a general velocity profile for all zero-pressure-gradient turbulent boundary-layer flows. To obtain c_f and a corresponding boundary-layer thickness δ for a given $Re_\delta*$ [or Re_θ—for hypersonic flow experiments, it has been argued that the calculated displacement thicknesses are found to be less sensitive to the choice of δ and thus of U_e; however, momentum thicknesses may sometimes be very sensitive (C. C. Horstman; private communication, Watson[9]) the following iterative procedure needs to be performed:

1) Given δ^* (or θ), guess δ^*/δ, θ/δ^* and u_τ (or θ/δ and u_τ).
2) Calculate $Re_{\delta_2} = \rho_e U_e \theta/\mu_w$ and find Π from Fig. 1a.
3) Calculate $y_\delta^+ = u_\tau \delta/\nu_w$ and obtain $U_{c,\delta}^+$ from Eq. (8).
4) Obtain the nontransformed dimensionless velocity U_δ^+ from Eq. (6).
5) Update $u_\tau = U_e/U_\delta^+$ and solve for $c_f = 2\ (T_e/T_w)(u_\tau/U_e)^2$.
6) Tabulate U as a function of η ($= y/\delta$) using Eqs. (8) and (6).
7) Update δ^*/δ and θ/δ^* (or θ/δ) by performing the following integrations numerically:

$$\frac{\theta}{\delta} = \int_0^1 \frac{\rho U}{\rho_e U_e}\left(1 - \frac{U}{U_e}\right) d\eta$$

$$\frac{\delta^*}{\delta} = \int_0^1 \left(1 - \frac{\rho U}{\rho_e U_e}\right) d\eta$$

where ρ/ρ_e is replaced by T_e/T with T obtained from Eq. (4). Steps 1 to 7 are repeated until the solution converges.

III. Results and Discussion

Figure 3 shows comparisons of skin friction at $Re_\theta = 10{,}000$ for an air flow over an insulated surface, for Mach numbers ranging from 0 to 10. At $M = 0$, both the present prediction and the Van Driest II formula reduce to the von Kármán-Schoenherr skin-friction formula. As Mach number increases, the present method predicts higher skin friction than the Van Driest II theory; at $M = 10$, it is nearly 13% higher. Since for a fixed Re_θ, the value of y^+ at the edge of the boundary layer y_δ^+ decreases rapidly as Mach number increases, comparisons are also presented for results obtained for a fixed y_δ^+ equal to

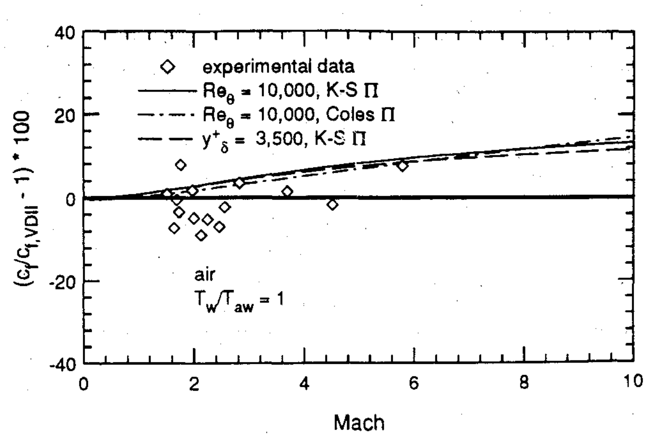


Fig. 3 Effect of Mach number on predictions of adiabatic-wall skin friction. Data from review by Hopkins and Inouye.[9]

3500, a value corresponding to that of an incompressible flow at $Re_\theta = 10{,}000$. As can be seen from the figure, the higher level of skin friction prediction is almost insensitive to the choice of Re_θ, indicating that the present prediction and the Van Driest theory behave in a similar fashion when subject to the change of Reynolds number. The skin-friction values obtained based on Coles' Π function, or Eq. (10), are also shown in the figure and they are almost the same as the ones obtained using the Π function defined by the von Kármán-Schoenherr skin-friction correlation. Finally, it is important to point out that the maximum Mach number for the data used in Hopkins and Inouye's[9] comparison is only about 6. These data are also presented in the figure for comparison, and one may argue that the present method is at least as good as the Van Driest II formula.

To further compare the present method and the Van Driest II theory in the hypersonic range, we use the data of Watson,[8] who measured skin friction and mean flow properties independently for helium flows with $M \approx 11$ at the boundary-layer edge (the title of the paper quotes a reference Mach number). In Watson's experiments, the surface was slightly heated with $T_w/T_{aw} \approx 1.05$. Figure 4a shows the effect of Mach number on predictions of skin friction at $Re_\theta = 10{,}000$ for helium in near adiabatic-wall conditions. Compared to the air flow results shown in Fig. 3, the helium flow results shown in Fig. 4a indicate a larger difference in the skin-friction values predicted using the two Π functions in high Mach numbers. The difference in the skin-friction predictions is also reflected in the comparisons of the local skin-friction and velocity profiles, shown in Figs. 4b and 4c, respectively. In Figs. 4b and 4c, the results were obtained by prescribing the experimental freestream conditions. Since Watson commented that his values of δ^* were more reliable than values of θ, the calculations were performed with the experimental δ^*. As shown in the figure, predictions of both skin-friction and velocity profiles using the Π function defined by the von Kármán-Schoenherr skin-friction correlation agree very well with the experiments. The disagreement of the skin-friction prediction at $Re_x = 21 \times 10^6$ is because the experiment at this Re_x is still laminar. On the other hand, the skin-friction values obtained by using Coles' Π function appear to be higher than the experimental data, and the velocity profiles obtained using Coles' Π function are also not as good as the ones obtained using the Π function defined by the von Kármán-Schoenherr skin-friction correlation. The Van Driest II prediction based on the experimentally reported Re_θ is also presented in the figure, and the results show too low a level of skin friction, similar to that in Fig. 4a.

Our predictions for nonadiabatic flat plates in air are presented in Fig. 5. The experimental data used by Hopkins and Inouye are also shown in the figure, with the experimental Mach numbers ranging from 3 to 7.5. To compare with the experimental data, the results are presented for three Mach numbers: 2, 5, and 10. Figure 5 shows in general that the present method predicts higher levels of c_f than those obtained by the Van Driest II theory at high T_w/T_{aw}, whereas at strongly cooled wall conditions the skin friction predicted by the present method is lower. The overprediction of the skin friction compared with Van Driest II theory for high T_w/T_{aw} has been discussed previously and is supported by Watson's experiments. The tendency of the present calculated skin friction to fall below the Van Driest II predictions in strongly cooled conditions also seems to be supported by the experimental data.

Hopkins et al.[13] have made some direct measurements of skin friction and velocity profiles for $M = 6$ to 7.8 at $T_w/T_{aw} = 0.3$ to 0.5 (see Fig. 6a) and concluded that the Van Driest II theory gave the most satisfactory skin friction predictions among all the theories they investigated. Figure 5 shows that the present method is almost identical to the Van Driest II theory in this range of wall temperature ratio. To further confirm the ability of the present method to predict velocity profiles on cooled flat plates, we have chosen Hopkins et al.'s[13] and Albertson and Ash's[14] experiments for compar-

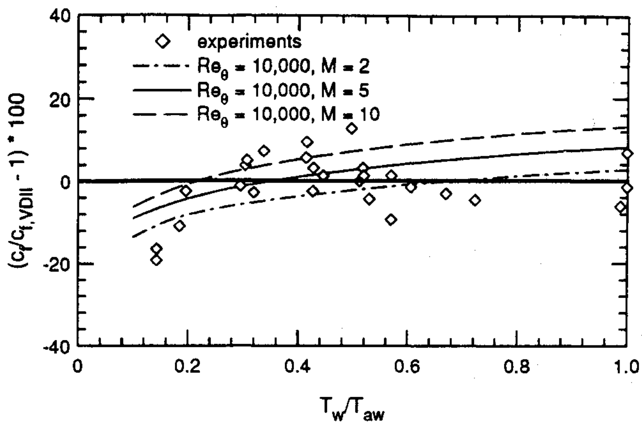


Fig. 5 Effect of wall-temperature ratio on predictions of skin friction. Data from review by Hopkins and Inouye.[9]

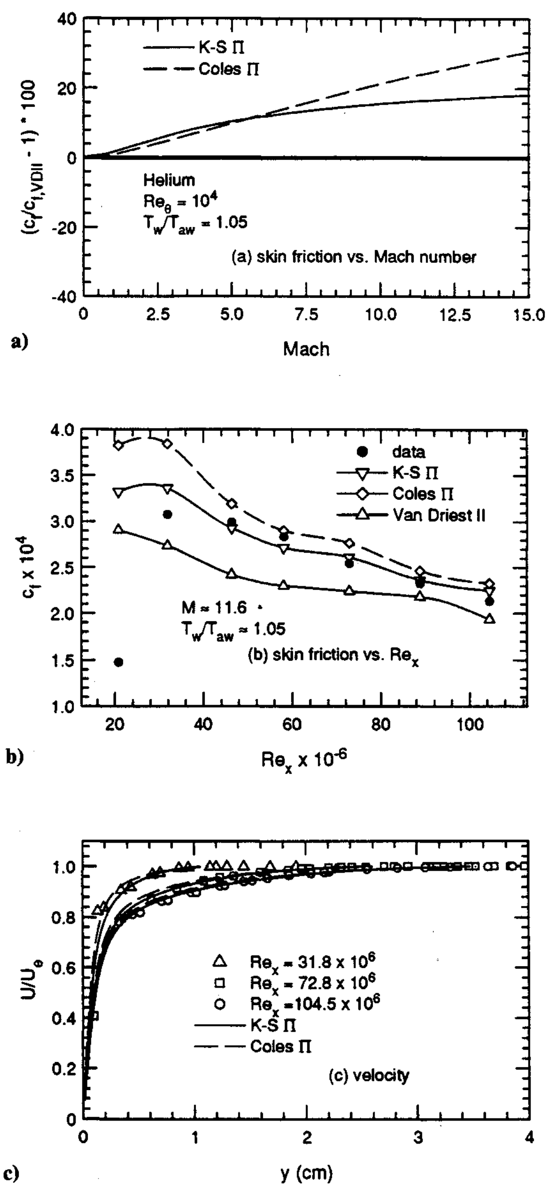


Fig. 4 Comparison with Watson's Mach 11 data: a) skin friction vs Mach number; b) skin friction vs Re_x; and c) velocity.

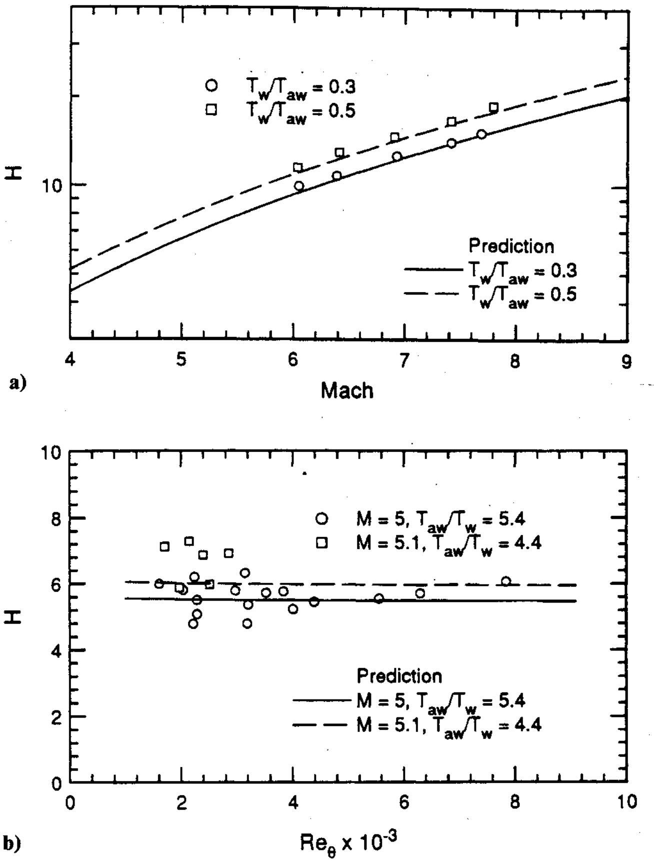


Fig. 6 Shape parameter H: comparison with a) data of Hopkins et al. and b) data of Albertson and Ash. (Here, Re_θ is evaluated from fluid properties at a reference temperature given by Albertson and Ash.[14])

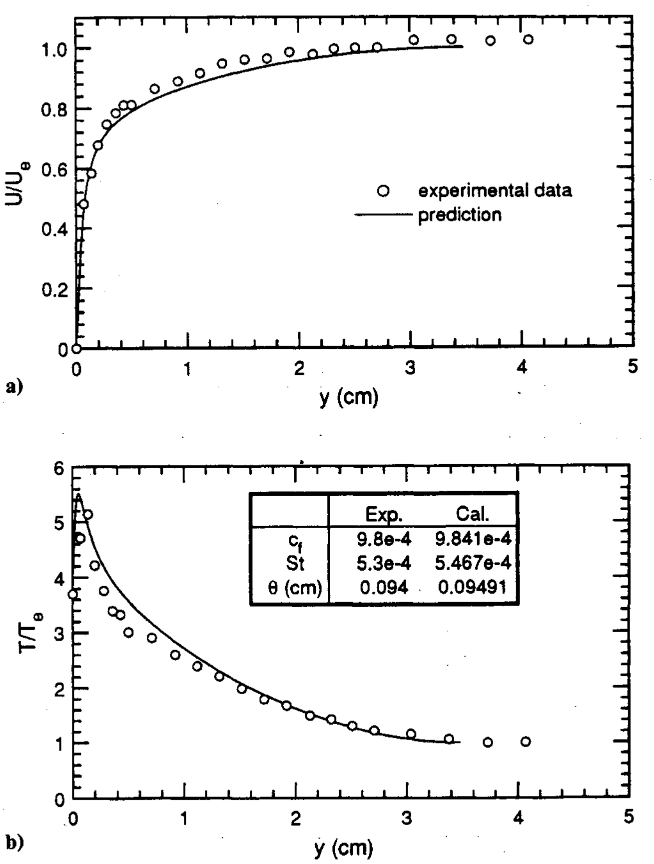


Fig. 7 Comparison with Kussoy and Horstman Mach 8.2 data.

isons. Albertson and Ash's experiment is for a Mach 5 flow with $T_w/T_{aw} \approx 0.18$. Under these flow conditions, shown in Fig. 5, one would expect to see differences in the skin-friction predictions between the present method and the Van Driest II theory. Unfortunately, they did not measure the skin friction, and the only relevant comparisons are with velocity profile shape. Figure 6 shows the comparison of the shape factors $H = \delta^*/\theta$ of the present method with the experimental values. Agreement with both experiments is excellent.

Last, a comparison with recent measurements by Kussoy and Horstman[15] is presented. The experiment is for a Mach 8.2 boundary layer in zero pressure gradient, with $T_w/T_{aw} \approx 0.28$. Both velocity and temperature are deduced from independent total and static pressure and total temperature surveys, and U_e is taken as the value of U at $y \approx 2.7$ cm. Figure 7 shows the comparison of the velocity and temperature profiles for $\delta^* = 1.59$ cm. Again, the comparison with the experimental data is very good. In addition, the present method successfully predicts the skin friction, the heat transfer Stanton number, and the momentum thickness, as shown in the panel of Fig. 7b.

Concluding Remarks

A self-consistent method to predict skin friction and velocity profiles of compressible boundary layers with zero pressure gradient is presented. The method has been shown to give excellent predictions when compared to experimental data. In the Mach number range of the data used in the early 1970s to establish the Van Driest II theory as the preferred method, the present method predicts skin friction nearly as well as Van Driest II. It has the advantage that low-Reynolds-number effects on the "wake" profile shape are taken into account. At higher Mach numbers on adiabatic walls, the present method predicts higher values of skin friction than Van Driest II and seems to agree better with recent Mach 11 helium flow data. On the other hand, the present skin friction predictions fall below Van Driest II on strongly cooled walls, which again gives a better fit to the data.

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